Mathematics offers us a very special way to look at the world. Other fields of study, from biology to history, from physics to psychology, from linguistics to palaeontology, look at how the world happens to be, and then try to come up with reasons for why it might be that way. Mathematics turns the idea that this how you gain understanding on its head. But what other way to discover things could there possibly be?

In mathematics, we begin by agreeing to assume some very basic statements are true, and discovery is all about figuring out what else must be true if those things are, and particularly the things that may surprise us.

For example, let's imagine it's true that: *When I'm in the rain, then I use an umbrella.* This would allow us to know many other things are true. Have a look at these two:

- When I'm in the rain <u>and</u> it's a Saturday, then I'm using an umbrella on a Saturday.
  *Not exactly exciting. The fact that it's a Saturday is completely unconnected to the rest of the statement. I've fused it with both parts, and the result is absolutely true, but rather dull.*
- When I'm not using an umbrella, I'm not in the rain.
  *There we go! I hope your first reaction was to think, "is this really true?" Remember, we only mind whether it's true given that our original assumption is known to be true. For a moment it tempts us to think there might be an exception somehow, but after a while we have to concede that yes, this follows from our assumption.*

These basic assumptions we start with were called ***postulates*** by the Ancient Greeks, and modern mathematicians like to call them ***axioms***, but knowing the words isn't as important as understanding the concept.

**<u>Definition</u>** An axiom is a statement which is defined to be true. A collection of axioms opens up a world to explore, the world in which none of those statements are ever wrong.

Because axioms are defined to be true no matter how silly they sound, you might think anything goes. But that's not quite true: in mathematics, entire worlds can collapse in an instant. In chapter 1 we will see this actually happened to the world that the Greek mathematician Pythagoras believed in. Another mathematician proved that the axioms he thought our world was built on actually lead to a contradiction, and so couldn't possibly be true. The world he had spent his life exploring turned out never to have existed. Poor Pythagoras.

Almost two thousand years later, it wasn't just a charismatic Greek guy and his followers who saw their world implode, but a global fraternity [1]of mathematicians who saw their area of mathematics, set theory – the foundation for pretty much all the rest – crack open, and paradoxes begin to spill out. Those were scary times to be a set theorist, until the hero of the decade, Bertrand Russell, arrived on the scene to repair the damage. To be fair, Russell was also the one to have pointed out the cracks, but no one blamed him for that: he could only point them out because they were there.

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[1] Yes, I'm afraid mathematics was almost exclusively for men back then.

For now, we won't go into the problems Russell uncovered in what we now call 'naïve set theory'. Ouch! Again, people had spent their lifetimes studying this stuff. The reason I brought him up here, is because of a wonderful story in which he warns about the dangers of assumptions.

Supposedly, Bertrand Russell was giving a lecture, and at some point said, "If you begin with an incorrect assumption, any falsehood can be proven." "Oh yeah?" some smart-ass from the audience shouted out, "so if $2 + 2 = 5$, you can prove you're the pope?" Mr Russell, taken aback by the question, but not exactly stumped by it, just smiled slightly, and explained, "Let us indeed imagine that $2 + 2$ is equal to $5$. It then follows, by subtracting on both sides, that $2 = 3$, and subtracting $1$ more we find that $1$ equals $2$. The pope and I are two people, therefore we are one person, therefore I am the pope." I imagine a substantial smattering of applause followed.

I don't mind knowing or admitting that this is a level of brilliance I don't imagine I'll ever be able to show off. I can still enjoy it, and that's one of the great things about mathematics. I can watch a chess grandmaster play, and be astounded, but even after seeing it I won't fully understand why all the moves made sense. But with Russell's train of thought, I may not have come up with it myself, but I can follow every step, and so I can enjoy it to the full. You'll see that a lot in this book. We will be proving how certain mathematical rules follow one from the other, and even though you didn't come up with it yourself, you will find you are able to understand how they do work, step by step, through ***logic***.

**Definition** Logic is a system for rephrasing or combining knowledge to learn new things which are just as certain as the things you already assumed were true.

Though some amazing logical thinking features in the story about Bertrand Russell, it isn't actually what the story is about. Logical thinking can lead you down blind alleyways if you begin with incorrect assumptions, and so you need to try to avoid this, by ***critical thinking***.

**Definition** Critical thinking is the act of doubting what you know and remaining open to changing your view of a mathematical or real world.

To be honest, I'm not sure whether the story about Bertrand Russell is true. I think it is, but there is a possibility that it is the result of someone wanting to show the power of logical thinking, the importance of critical thinking, and them inventing a suitable story using creative thinking.

**Definition** Creative thinking is the act of coming up with useful or interesting new ideas.

These ways of thinking are, conceptually, the only tools we'll need to explore the worlds we can create with axioms. Of course, we will be discovering new interesting truths as we explore them, and these statements will become tools in their own rights. We'll be giving these statements names whose fanciness will rival that of our axioms: ***rules***, ***lemma's***, ***theorems*** and occasionally even ***fundamental theorems***. Again, you do not need to know those words.

We'll be discovering these new truths by proving them.

**Definition** A proof of a statement is a logical process which shows how that statement is true or false based on its connection to other statements which are already known to be true.